Envelope Solitons of Nonlinear Schrödinger Equation with an Anti-cubic Nonlinearity[1]

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Abstract

On the basis of a recently-proposed method to find solitary solutions of generalized nonlinear Schrödinger equations [1]-[3], the existence of an envelope solitonlike solutions of a nonlinear Schrödinger equation containing an anti-cubic nonlinearity ($|\Psi|^{-4}\Psi$) plus a "regular" nonlinear part is investigated. In particular, in case the regular nonlinear part consists of a sum of a cubic and a quintic nonlinearities (i.e. $q_1|\Psi|^2\Psi + q_2|\Psi|^4\Psi$), an upper-shifted bright envelope solitonlike solution is explicitly found.

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Let us consider the following generalized Korteweg-de Vries equation (GKdVE)

$$a\,\frac{\partial u}{\partial s}\ -\ G\,[u]\ \frac{\partial u}{\partial x}\ +\ \frac{\nu^2}{4}\ \frac{\partial^3 u}{\partial x^3}\ =\ 0\quad , \tag{1}$$

where a and ν are real constants, and $G\,[u]$ is a real functional of u, and the following generalized nonlinear Schrödinger equation (GNLSE)

$$i\alpha\frac{\partial \Psi}{\partial s}\ +\ \frac{\alpha^2}{2}\frac{\partial^2 \Psi}{\partial x^2}\ -\ U\left[|\Psi|^2\right]\Psi\ =\ 0\quad , \tag{2}$$

where $U\,[|\Psi|^2]$ is a real functional of $|\Psi|^2$ and α is a real constant.
Recently, it has been shown that a correspondence between (1) and (2) has been constructed in such a way that, provided that the following equations are satisfied, i.e.

$$\nu\ =\ \alpha \tag{3}$$

$$u_0 a\ =\ -\,u_0^2\ +\ 2c_0 \tag{4}$$

$$G\,[u]\ =\ u\frac{dU\,[u]}{du}\ +\ 2U\,[u]\quad , \tag{5}$$

if $u(x-u_0 s)\equiv u(\xi)$ (u_0 being a real constant) is a non–negative stationary–profile solution of (1) thus

$$\Psi(x,s)=\sqrt{u(\xi)}\exp\left\{\frac{i}{\alpha}\left[\phi_0-\left(c_0+u_0^2\right)s+u_0 x+A_0\int\frac{d\xi}{u(\xi)}\right]\right\}\quad , \tag{6}$$

is a stationary–profile envelope solution of (2), where ϕ_0, c_0, and A_0 are real constants (not all independent) [2]. Note that $u(\xi)=|\Psi(x,s)|^2$. This result has been successfully applied to find new analytical solitary-wave envelope solutions of some modified nonlinear Schrödinger equations (MNLSE) containing high–order nonlinearities (with respect to the standard cubic nonlinearity) [2, 3]. In particular, analytical solutions in the form of bright and dark/grey envelope solitonlike solutions have been found for a MNLSE with cubic plus quintic nonlinear terms, i.e.

$$\mathcal{U}\left(|\Psi|^2\right)=q_1|\Psi|^2+q_2|\Psi|^4\quad . \tag{7}$$

In fact, it has been shown that the following MNLSE

$$i\alpha\frac{\partial \Psi}{\partial s}\ +\ \frac{\alpha^2}{2}\frac{\partial^2 \Psi}{\partial x^2}\ -\ \left[q_1\ |\Psi|^2\ +\ q_2\ |\Psi|^4\right]\Psi\ =\ 0\quad , \tag{8}$$

has the following envelope solitonlike solutions [2]:

$$\begin{aligned}\Psi(x,s)\ &=\ \sqrt{\overline{u}\ \left[1\ +\ \epsilon\ \mathrm{sech}\,(\xi/\Delta)\right]}\exp\left\{\frac{i}{\alpha}\left[\phi_0-As+u_0x\right]\right\}\times\\ &\times\ \exp\left\{\frac{iB}{\alpha}\left[\frac{\xi}{\Delta}+\frac{2\epsilon}{\sqrt{1-\epsilon^2}}\arctan\left(\frac{(\epsilon-1)\tanh\,(\xi/2\Delta)}{\sqrt{1-\epsilon^2}}\right)\right]\right\}\quad ,\end{aligned} \tag{9}$$

where ϕ_0 still plays the role of arbitrary constant,

$$\epsilon = \pm\sqrt{1 - 32|q_2|\left(u_0 - V_0\right)^2 / \left(3q_1^2\right)} \quad ,$$

$$\Delta = |\alpha| / \left(2\sqrt{2|E_0'|}\right) \quad ,$$

$$E_0' = -3q_1^2/\left(64|q_2|\right) \ + \ \left(u_0 - V_0\right)^2/2 \quad ,$$

provided that $E_0' < 0$ and $q_2 < 0$ and

$$-\sqrt{\frac{3q_1^2}{32|q_2|}} + V_0 < u_0 < \sqrt{\frac{3q_1^2}{32|q_2|}} + V_0 \quad ,$$

$$A \ = \ \frac{15q_1^2}{64|q_2|} + \frac{\left(u_0 - V_0\right)^2}{2} + \frac{u_0^2}{2} \quad , \tag{10}$$

and

$$B \ = -\frac{|\alpha|\left(u_0 - V_0\right)}{2\sqrt{2|E_0'|}} \quad . \tag{11}$$

The following four cases have been discussed [2]

(a). $0 < \epsilon < 1$ $(u_0 - V_0 \neq 0)$:

$$u(\xi = 0) = (1+\epsilon)\overline{u} \ , \quad \text{and} \quad \lim_{\xi\to\pm\infty} u(\xi) = \overline{u}$$

which corresponds to a bright soliton of maximum amplitude $(1+\epsilon)\overline{u}$ and up-shifted by the quantity $\overline{u}$ (*up-shifted bright soliton*).

(b). $-1 < \epsilon < 0$ $(u_0 - V_0 \neq 0)$:

$$u(\xi = 0) = (1-\epsilon)\overline{u} \ , \quad \text{and} \quad \lim_{\xi\to\pm\infty} u(\xi) = \overline{u}$$

which is a dark soliton with minimum amplitude $(1-\epsilon)\overline{u}$ and reaching asimptotically the upper limit $\overline{u}$ (standard *gray soliton*).

(c). $\epsilon = 1$ $(u_0 - V_0 = 0)$:

$$u(\xi = 0) = 2\overline{u} \ , \quad \text{and} \quad \lim_{\xi\to\pm\infty} u(\xi) = \overline{u}$$

which corresponds to a bright soliton of maximum amplitude $2\overline{u}$ and up-shifted by the maximum quantity $\overline{u}$ (*upper-shifted bright soliton*).

(d). $\epsilon = -1$ $(u_0 - V_0 = 0)$:

$$u(\xi = 0) = 0 \ , \quad \text{and} \quad \lim_{\xi\to\pm\infty} u(\xi) = \overline{u}$$

which is a dark soliton (zero minimum amplitude), reaching asimptotically the upper limit $\overline{u}$ (standard *dark soliton*).

In this paper, we use the method mentioned above to find envelope solitonlike solutions of the following MNLSE containing, besides the cubic and quintic nolinearities, an "anti-cubic" nonlinearity (i.e. $|\Psi|^{-4}\Psi$), namely

$$i\alpha\frac{\partial\Psi}{\partial s} + \frac{\alpha^2}{2}\frac{\partial^2\Psi}{\partial x^2} - \left[Q_0\,|\Psi|^{-4} + q_1\,|\Psi|^2 + q_2\,|\Psi|^4\right]\Psi = 0 \quad , \tag{12}$$

where Q_0 is a real constant.
It is easily seen that Eq. (5) has the following general solution

$$U\,[u] = \frac{1}{u^2}\left[K_0 + \int G\,[u]\;u\;du\right], \tag{13}$$

where K_0 is an arbitrary real constant. However, once the function $u(\xi)$ is a stationary profile of (1) with

$$G\,[u] = 3q_1\,u + 4q_2\,u^2\,, \tag{14}$$

it follows that, provided that $Q_0 = K_0$,

$$\mathcal{U}\,[u] = \frac{1}{u^2}\int G\,[u]\;u\;du\,, \tag{15}$$

where $\mathcal{U}$ is defined by (7). Consequently, it seems that the mapping (6) allows us to construct solutions of the (12) which have the functional form of the ones satisfying the (8). The only restriction for (12) is that $u(\xi)$ must not vanish somewhere, which implies that we have to impose that $u(\xi)$ be a positive solitonlike solution of the MKdVE of the type (1) with the nonlinearity given by (14). In fact, the vanishing of u corresponds to a divergence of the nonlinear potential term $|\Psi|^{-4}$ in (12). This circumstance excludes the standard dark solitonlike solutions.
On the other hand, it can be shown that [2]

$$-\frac{\nu^2}{2}\frac{d^2u^{1/2}}{d\xi^2} + \frac{Q_0}{u^{3/2}} + \frac{1}{u^{3/2}}\int G\,[u]\;u\;du = \left(c_0 + \frac{u_0^2}{2}\right)u^{1/2} - \frac{A_0^2}{2u^{3/2}} \quad . \tag{16}$$

It is then clear from (16) that a family of solitary wave solutions of (12) can be obtained by imposing the following condition

$$A_0 = \pm\sqrt{-2Q_0}\,, \tag{17}$$

which implies that such a kind of family of solution exists for negative values of Q_0. Consequently, Eq. (16) becomes the following NLSE for stationary states

$$-\frac{\alpha^2}{2}\frac{d^2u^{1/2}}{d\xi^2} + \mathcal{U}\,[u]\,u^{1/2} = E_0 u^{1/2} \quad , \tag{18}$$

where $E_0 = c_0 + u_0^2/2$. This equation admits solitary solutions whose form is the squared modulus of (9). It follows that for any

$$Q_0 < 0 \quad , \tag{19}$$

under condition (17), solitary-wave solutions of (12) can be directly constructed from both (6) and (9) as

$$\Psi_{\pm}(x,s) = \sqrt{\overline{u}\ \left[1\ +\ \epsilon\ \mathrm{sech}\left(\frac{\xi}{\Delta}\right)\right]}\ \times$$
$$\times\ \ \exp\left\{\frac{i}{\alpha}\left[\phi_0 - \left(E_0 + \frac{u_0^2}{2}\right)s + u_0 x \pm\sqrt{2|Q_0|}\ \int\frac{d\xi}{\sqrt{\overline{u}\ \left[1\ +\ \epsilon\ \mathrm{sech}\left(\xi/\Delta\right)\right]}}\right]\right\}, \quad (20)$$

where, in principle, according to Ref. [2], ϵ should be taken in the following range

$$-1\ < \epsilon\ \leq 1 \quad , \quad (21)$$

which excludes the standard "dark" solitary waves ($\epsilon = -1$), namely the condition for which the modulus of Ψ vanishes at $\xi = 0$.
Actually, the direct substitution of $u = |\Psi|^2$ given by (20) into the eigenvalue equation (18) allows us to find

$$\epsilon\ =\ 1 \quad , \quad (22)$$

$$\overline{u}\ =\ -\frac{3q_1}{8q_2} \quad , \quad (23)$$

$$q_1\ >\ 0 \quad , \qquad q_2\ <\ 0 \quad , \quad (24)$$

$$E_0\ =\ -\frac{15q_1^2}{64q_2} \quad , \quad (25)$$

$$\Delta = \frac{2|\alpha|}{q_1}\sqrt{\frac{2|q_2|}{3}} \quad . \quad (26)$$

Consequently, solution (20) can be cast as

$$\Psi_{\pm}(x,s) = \sqrt{\frac{3q_1}{8|q_2|}\ \left[1\ +\ \mathrm{sech}\left(\frac{\xi}{\Delta}\right)\right]}\ \times$$
$$\times\ \ \exp\left\{\frac{i}{\alpha}\left[\phi_0 - \left(\frac{15q_1^2}{64|q_2|} + \frac{u_0^2}{2}\right)s + u_0 x \pm\sqrt{2|Q_0|}\ \frac{16|\alpha||q_2|}{3q_1^2}\sqrt{\frac{2|q_2|}{3}}\left(\frac{\xi}{\Delta} - \tanh\left(\frac{\xi}{2\Delta}\right)\right)\right]\right\} \quad , \quad (27)$$

where u_0 is a fully arbitrary soliton velocity. According to the classification of the solitary waves given in Ref. [2], the (27) represents an upper-shifted bright envelope solitonlike solution of Eq. (12), provided that the coefficients Q_0, q_1 and q_2 satisfy the conditions (19) and (24), respectively. It is clear that Eq. (22), which does not contradict condition (21), implies that also gray solitary solutions do not exist in the solution form (20).
In conclusion, in this paper, by using a recently developed method for solving a wide family of MNLSE with high-order nonlinearities [1]-[3] on the basis of the knowledge of the solution of the associated MKdVE, an upper-shifted bright envelope solitonlike solution of a MNLSE, containing, besides the standard cubic and quintitic nonlinearities, an "anti-cubic" nonlinear term (see Eq. (12)). Our analysis has shown that, once the form of the envelope solution of (12) is taken according to (20), dark and gray envelope solitary solutions do not exist that, in contrast, exist in case the "anti-cubic" term is missing.

References

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